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The ten classical types of group representations

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Abstract. Each irreducible n-dimensional complex representation of an arbitrary group $\mathscr{G}$ is shown to be associated with a unique classical group. Such representations are thereby classified into 'ten classical types' associated with the respective classical groups $\mathrm{GL}(n;\mathbb{C})$, $\mathrm{O}(n;\mathbb{C})$, $\mathrm{Sp}(n;\mathbb{C})$, $\mathrm{U}(r,s)$, $\mathrm{GL}(n;\mathbb{R})$, $\mathrm{GL}(n/2;\mathbb{H})$, $\mathrm{O}(r,s)$, $\mathrm{Sp}(n;\mathbb{R})$, $\mathrm{O}(n/2;\mathbb{H})$, $\mathrm{Sp}(r/2,s/2)$, where $r+s=n$.

1. Introduction

Finite-dimensional complex vector spaces present themselves in quartets of the kind

$$V, V', \bar{V}, \bar{V}' \tag{1}$$

(see Shaw (1982), § 1.5.5). Here V' denotes the *dual*, $\bar{V}$ the *antispace* and $\bar{V}'$ the *antidual* of the complex vector space V. We can consider $\bar{V}'$ as antispace $\overline{V'}$ of the dual of V, or as dual $(\bar{V})'$ of the antispace of V, or as the space $AL(V,\mathbb{C})$ of antilinear functionals defined on V. In fact, complete democracy reigns within a quartet: for example, starting out from V' we can consider it to have dual V, antispace $\bar{V}'$ and antidual $\bar{V}$.

Consequently finite-dimensional complex representations of a group $\mathscr{G}$ come along in corresponding quartets

$$D, \hat{D}, \bar{D}, \hat{\bar{D}} \tag{2}$$

whose respective carrier spaces are as in (1). Here $\hat{D}$ denotes the contragredient representation, $\bar{D}$ denotes the complex conjugate representation and $\hat{\bar{D}}$ denotes the contragredient of $\bar{D}$, or equally well the complex conjugate of $\hat{D}$. If one of the quartet (2) is an irreducible representation, then so of course are the other three. From now on we restrict our attention to irreducible representations.

Given such a quartet of representations of $\mathscr{G}$ we can immediately distinguish the following cases.

Case O. All four representations are (linearly) inequivalent.

$$\left.\begin{array}{l}\textit{Case I.}\ D\cong\hat{D},\ \bar{D}\cong\hat{\bar{D}}\\ \textit{Case II.}\ D\cong\hat{\bar{D}},\ \hat{D}\cong\bar{D}\\ \textit{Case III.}\ D\cong\bar{D},\ \hat{D}\cong\hat{\bar{D}}\end{array}\right\}\ \text{but no other equivalences occur.} \tag{3}$$

Case IV. All four representations are (linearly) equivalent.

Actually, instead of the linear equivalence (in cases II and IV) of D and $\hat{\bar{D}}$ we prefer from now on to consider the antilinear equivalence of D and $\hat{D}$, and instead of the linear equivalence (in cases III and IV) of D and $\bar{D}$ we prefer to consider the antilinear self-equivalence of D. Thus cases I, II and III are characterised by the existence of one, *and only one*, of the linear or antilinear isomorphisms G, P and K which satisfy, respectively, the following intertwining properties (4), (5) and (6):

Case I. There exists a linear isomorphism $G: V \to V'$ which intertwines D with $\hat{D}$:

$$GD(g) = \hat{D}(g)G \qquad \text{for all } g \in \mathscr{G}. \tag{4}$$

Case II. There exists an antilinear isomorphism $P: V \to V'$ which intertwines D with $\hat{D}$:

$$PD(g) = \hat{D}(g)P \qquad \text{for all } g \in \mathscr{G}. \tag{5}$$

Case III. There exists an antilinear isomorphism $K: V \to V$ which commutes with the $D(g)$:

$$KD(g) = D(g)K \qquad \text{for all } g \in \mathscr{G}. \tag{6}$$

Correspondingly, case IV is characterised by the simultaneous existence of two, and hence all three, of the above isomorphisms G, P and K.

As is spelled out in the next section, case I is equally well discussed in terms of the existence of a non-degenerate invariant bilinear form, and divides into two subcases, I_+ and I_-, according to whether the bilinear form is symmetric or skew symmetric. Similarly case II is equally well discussed in terms of the existence of a non-degenerate invariant sesquilinear form; this last can be taken to be Hermitian, and so case II divides into subcases $\mathrm{II}^{(r,s)}$ according to the signature (r, s), $r+s=n=\dim V$, of the form. Furthermore case III divides into two subcases, III_+ and III_-, according to whether the representation D is of real or quaternionic type.

Consequently case IV divides into subcases $\mathrm{IV}^{(r,s)}_{\varepsilon\eta}$ (combining together III_ε, $\mathrm{II}^{(r,s)}$, I_η). Here all four possibilities $++$, $+-$, $-+$, $--$ for ε, η can be realised, but for the last three of these possibilities we will see that not all values of the signature (r, s) are allowed.

As will be summarised in table 1, the result is that the representation D is uniquely associated with a certain classical group, and so can be assigned to just one of 'ten classical types' of representation. Three of these ten types subdivide further according to the value of the signature (r, s).

The classical groups themselves are of course very well known. The ones occurring in this present work, and in most of the theoretical physics literature, are those associated with vector spaces over $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$ (where $\mathbb{H}$ denotes the non-commutative division algebra of the real quaternions). In addition to the general linear groups of such spaces, the other classical groups arise (see, for example, Birkoff and Von Neumann 1936, Dieudonné 1971, Porteous 1969) from consideration of the isometries of those (non-degenerate sesquilinear) scalar products which are *orthosymmetric*, i.e. are such that $x \cdot y = 0$ holds if and only if $y \cdot x = 0$ holds. Mathematical physicists usually become acquainted with them via texts such as Helgason (1978) or Barut and Raczka (1977), where they are used to provide realisations, at the Lie group level, for each of Cartan's non-exceptional simple real Lie algebras.

The chief point of the present work is to demonstrate that all of these classical groups also arise very naturally from a classification of finite-dimensional irreducible complex representations of *arbitrary* groups. As will be seen below, the derivation of this classification scheme requires only a rudimentary knowledge of group representations, involving very little more than appeals to Schur's lemma. Furthermore, it should be pointed out that the scheme is in part already well known. In particular (6) is treated in most texts on group representations, and the representation D is thereby assigned to one of three Frobenius-Schur types:

$$(0) = \text{strictly complex} \qquad (+) = \text{real} \qquad (-) = \text{quaternionic}$$

according as (6) has a solution K which

type (0): is of necessity $K = 0$

type (+): satisfies $K^2 = +I$

type (−): satisfies $K^2 = -I$.

(In the classification of the present paper these Frobenius-Schur types subdivide further—see (40).)

It may well be the case that the content of table 1 is known to many experts in the field, and may even be implicit in the published work of some; however, the author is not aware of any explicit detailed treatment in the standard literature concerning case IV of table 1.

2. The ten classical types

Let $n = \dim V$, where V carries the (irreducible, complex) representation D of the group $\mathscr{G}$.

Case I. If $G \in L(V, V')$ satisfies (4), then so does its transpose $G^t \in L(V, V')$. (Strictly of course $G^t \in L(V'', V')$, but we make the customary identification (see, for example, Shaw (1982), (1.38)) of the double dual V'' of V with V itself.) But by Schur's lemma any non-zero solution $G \in L(V, V')$ of (4) is not only of necessity an isomorphism but is also unique up to a scalar multiple. Consequently $G^t = \eta G$, where $\eta \in \mathbb{C}$ satisfies $\eta^2 = 1$ on account of the property $(G^t)^t = G$. Thus case I divides into two subcases, according to whether $\eta = +1$ or $\eta = -1$:

$$\textit{Case } I_+.\ G^t = G \qquad \textit{Case } I_-.\ G^t = -G. \tag{7}$$

Upon defining the bilinear form $x \cdot y$ on V by

$$x \cdot y = \langle x, Gy \rangle \qquad x\,y \in V \tag{8}$$

where $\langle\ ,\ \rangle: V \times V' \to \mathbb{C}$ denotes the natural pairing of V and V', we see that the intertwining property (4) translates into the invariance property

$$D(g)x \cdot D(g)y = x \cdot y \qquad \text{for all } g \in \mathscr{G} \tag{9}$$

and the conditions (7) translate into the symmetry conditions:

$$\textit{Case } I_+.\ x \cdot y = y \cdot x \qquad \textit{Case } I_-.\ x \cdot y = -y \cdot x. \tag{10}$$

Thus V is equipped with an orthogonal (case I_+) or symplectic (case I_-) geometry which is (since G is an isomorphism) non-degenerate, and the image of $\mathscr{G}$ under D is restricted to lie inside the associated isometry subgroup of $\mathrm{GL}(V) \cong \mathrm{GL}(n; \mathbb{C})$:

$$\textit{Case } I_+.\ D(g) \in \mathrm{O}(V) \cong \mathrm{O}(n; \mathbb{C}) \qquad \textit{Case } I_-.\ D(g) \in \mathrm{Sp}(V) \cong \mathrm{Sp}(n; \mathbb{C}) \tag{11}$$

where in the symplectic case $n = 2m$ is necessarily even. Hence the entries under case I in the last two columns of table 1.

Case II. This is similar to case I, but because the mapping $P: V \to V'$ which satisfies (5) is antilinear, we need to appeal to a corresponding antilinear version of Schur's lemma (see, for example, Shaw (1982), exercise 3.4.3) in order to deduce that P is not only, if non-zero, of necessity an anti-isomorphism, but satisfies $P^{t} = \lambda P$. Since $(P^{t})^{t} = P$ and $(\lambda P)^{t} = \bar{\lambda} P^{t}$ we see that $\lambda \in \mathbb{C}$ satisfies $\bar{\lambda}\lambda = 1$. Hence by multiplying our original solution P of (5) by a suitable phase factor we can arrange for our new solution to satisfy $P^{t} = P$.

Upon defining the non-degenerate sesquilinear form (x, y) (linear in x, antilinear in y) on V by

$$(x, y) = \langle x, Py \rangle \tag{12}$$

we see that the intertwining property (5) translates into the invariance property

$$(D(g)x, D(g)y) = (x, y) \qquad \text{for all } g \in \mathscr{G} \tag{13}$$

and the condition $P^{t} = P$ translates into the condition $\overline{(x, y)} = (y, x)$ of Hermitian symmetry. Thus V is equipped with a non-degenerate (pseudo-)unitary geometry and the image of $\mathscr{G}$ under D is restricted to lie inside the associated isometry group:

$$\textit{Case II.}\ \ D(g) \in \mathrm{U}(V) \cong \mathrm{U}(r, s). \tag{14}$$

Here (r, s), $r + s = n$, denotes the signature of the Hermitian form (12). We can, if necessary, replace P by $-P$ so as to arrange $r \geqslant s$. In this way we assign a unique classical group $\mathrm{U}(r, s)$ to a representation belonging to case II.

Case III. If the non-zero antilinear operator $K: V \to V$ satisfies (6) then Schur's lemma (antilinear version) asserts not only that K has an inverse but also, since K^{-1} satisfies (6), that $K^{-1} = \varepsilon K$. Here, using $KK^{2} = K^{2}K$, we see that ε is necessarily real, and so by a suitable normalisation of K we can arrange for ε to be $+1$ or -1: $K^{2} = \pm I$ (but where the sign is *not* under our control). Consequently case III divides into two subcases:

$$\textit{Case III}_{+}.\ \ K^{2} = +I \qquad \textit{Case III}_{-}.\ \ K^{2} = -I. \tag{15}$$

In case III_{+}, K is a *conjugation* and the space V is the *complexification* $V = W^{\mathbb{C}}$ of the (n-dimensional) real vector space

$$W = \{x \in V : Kx = x\} \tag{16}$$

(cf Shaw (1982), §§ 1.5.5, 1.5.6). Since the operators $D(g) \in \mathrm{GL}(V) \cong \mathrm{GL}(n; \mathbb{C})$ commute with the conjugation K, and hence leave W invariant, they are complexifications of corresponding operators $A(g) \in \mathrm{GL}(W)$:

$$\textit{Case III}_{+}.\ \ D(g) = A(g)^{\mathbb{C}} \qquad \text{where } A(g) \in \mathrm{GL}(W) \cong \mathrm{GL}(n; \mathbb{R}). \tag{17}$$

In other words, if a(ny) basis for W is chosen as our basis for V, then the operators $D(g)$ are realised as real matrices $\in \mathrm{GL}(n; \mathbb{R})$.

In case III_{-}, K is a *quaternionic unit* and the space V is the *symplectification* $V = W^{\mathbb{C}}$ of a quaternionic vector space W. Here W coincides as a set with V, but is given the structure of a quaternionic space by laying down that K should play the role of the third quaternionic unit $k \in \mathbb{H}$ (which construction succeeds since $K(\mathrm{i}I) = -(\mathrm{i}I)K$, $\mathrm{i} = \sqrt{-1} \in \mathbb{C}$, mirrors the quaternionic relation $k\mathrm{i} = -\mathrm{i}k$). Thus for $q = \lambda + \mu k \in \mathbb{H}$ ($\lambda = \lambda_1 + \mathrm{i}\lambda_2 \in \mathbb{C}$, $\mu = \mu_1 + \mathrm{i}\mu_2 \in \mathbb{C}$), we define

$$qx = \lambda x + \mu Kx \tag{18}$$

and so convert V into a quaternionic space W. If $\{e_1, e_2, \ldots, e_m\}$ is a basis for W, then $\{e_1, Ke_1, \ldots, e_m, Ke_m\}$ is seen to be a basis for V. Consequently in case III$_-$ the dimension of V is necessarily even: $n = 2m$.

Let $A(g): W \to W$ denote the operator $D(g): V \to V$ when we switch our point of view and regard, as above, the $\mathbb{C}$ space V as the $\mathbb{H}$ space W. Since the $D(g)$ are not only $\mathbb{C}$ linear operations on V but also commute with K, it follows from the definition (18) that *the $A(g)$ are $\mathbb{H}$ linear operators on W*: $A(g)qx = qA(g)x$. We will say that the operators $D(g) \in \mathrm{GL}(V) \cong \mathrm{GL}(n; \mathbb{C})$ are symplectifications $A(g)^{\mathbb{C}}$ of corresponding operators $A(g) \in \mathrm{GL}(W)$:

$$\textit{Case III}_-.\ D(g) = A(g)^{\mathbb{C}} \qquad \text{where } A(g) \in \mathrm{GL}(W) \cong \mathrm{GL}(n/2; \mathbb{H}). \tag{19}$$

Thus in case III we are essentially dealing with an irreducible representation A carried by a real n-dimensional (case III$_+$) or quaternionic $n/2$-dimensional (case III$_-$) space W, where $W^{\mathbb{C}} = V$. For simplicity we will accordingly say that D is real in case III$_+$ and quaternionic in case III$_-$.

Case IV. The considerations invoked above to deal with cases I, II and III were elementary and essentially well known. Almost as elementary, but possibly not so well known, are the corresponding considerations required to treat a representation D belonging to case IV. *In this case all three of the foregoing 'structures' are simultaneously present*, namely:

(i) an invariant bilinear form $x \cdot y = \eta y \cdot x$ ($\eta = \pm 1$);

(ii) an invariant Hermitian form (x, y);

(iii) a structure map† K, satisfying $K^2 = \varepsilon I$ ($\varepsilon = \pm 1$).

Moreover these three structures are related: any two essentially determine the third. For P and GK both intertwine D with $\hat{D}$, and so, by Schur's lemma (antilinear version) once more, P must be a scalar multiple of GK. By replacing G by a suitable scalar multiple we can arrange that

$$P = \begin{cases} GK & \text{if } \varepsilon\eta = 1 \\ \mathrm{i}GK & \text{if } \varepsilon\eta = -1. \end{cases} \tag{20}$$

In other words we arrange that the three structures are related by

$$(x, y) = \begin{cases} x \cdot Ky & \text{if } \varepsilon\eta = 1 \\ \mathrm{i}x \cdot Ky & \text{if } \varepsilon\eta = -1. \end{cases} \tag{21}$$

Here G, P and K still satisfy

$$G^{\mathrm{t}} = \eta G \qquad P^{\mathrm{t}} = P \qquad K^2 = \varepsilon I \tag{22}$$

and furthermore P determines a signature (r, s) with $r + s = n$ and $r \geqslant s$. Observe that G, P and K are still arbitrary to the extent

$$G \mapsto \alpha\bar{\omega}G \qquad P \mapsto \alpha P \qquad K \mapsto \omega K \tag{23}$$

where $|\omega| = 1$ and α is real (and, in fact, positive if we abide by our arrangement $r \geqslant s$). Let K^+ and $\tilde{K}$ denote adjoints of K with respect to the Hermitian form (x, y) and bilinear form $x \cdot y$:

$$(K^+x, y) = \overline{(x, Ky)} \qquad \tilde{K}x \cdot y = \overline{x \cdot Ky}. \tag{24}$$

† cf Adams (1969), definition 3.2.

Then it follows from (21) and (22) that

$$K^{+}K = \varepsilon\eta I \tag{25a}$$

$$\tilde{K}K = I. \tag{25b}$$

To prove (25b), first of all let us see what happens if we omit the i in (21). Then we would obtain

$$Kx \cdot Ky = (Kx, y) = \overline{(y, Kx)} = \overline{y \cdot K^2 x} = \varepsilon\eta \overline{x \cdot y}$$

and hence we would obtain the result $\tilde{K}K = \varepsilon\eta I$.

Upon inserting the i in (21) in the cases when $\varepsilon\eta = -1$, we obtain the desired result $\tilde{K}K = I$, which last proves the more convenient result for our further deliberations. Hence our choice of arrangement in (21). The result (25a) is proved similarly (and, of course, is not affected by replacing K by iK).

Case IV divides into various subcases $\mathrm{IV}_{\varepsilon\eta}^{(r,s)}$ according to the values of the signs ε, η and of the signature (r, s). It should be noted that *for the cases when* $\varepsilon\eta = -1$ *the signature is necessarily neutral.* In these cases the operator K satisfies $K^{+}K = -I$ and so maps an orthonormal basis of signature (r, s) onto one of signature (s, r), whence by Sylvester's law of inertia $r = s = n/2$. So we need to consider cases $\mathrm{IV}_{++}^{(r,s)}$, IV_{+-}, IV_{-+} and $\mathrm{IV}_{--}^{(r,s)}$. (We will see in a moment that any signature is possible if $\varepsilon = \eta = +1$ but that r and s have to be even if $\varepsilon = \eta = -1$.)

In the cases $\varepsilon = +1$, when K is a conjugation, recall that $V = W^{\mathbb{C}}$ where the real n-dimensional vector space W consists (see (16)) of those elements of V which are 'real' with respect to K. Note that the restriction of the bilinear form $x \cdot y$ to $W \times W$ is real:

$$x \cdot y = Kx \cdot Ky = \overline{x \cdot y} \qquad \text{for } x \in W,\ y \in W$$

where the last equality follows from (25b). Thus in case $\mathrm{IV}_{++}^{(r,s)}$ W carries a real orthogonal geometry whose signature must be (r, s) since $x \cdot y = (x, y)$ for $x, y \in W$, while in case IV_{+-} W carries a (real) symplectic geometry. In the cases $\varepsilon = +1$ it thus follows from (9) and (17) that the $D(g)$ are complexifications of operators $A(g)$ lying inside the isometry group of the real vector space W:

$$\textit{Case IV}_{++}^{(r,s)}.\ D(g) = A(g)^{\mathbb{C}}\colon A(g) \in \mathrm{O}(W) \cong \mathrm{O}(r,s) \tag{26}$$

$$\textit{Case IV}_{+-}.\ D(g) = A(g)^{\mathbb{C}}\colon A(g) \in \mathrm{Sp}(W) \cong \mathrm{Sp}(n, \mathbb{R}). \tag{27}$$

In the cases $\varepsilon = -1$ when K is a quaternionic unit recall that $V = W^{\mathbb{C}}$ where the quaternionic m-dimensional $(m = n/2)$ vector space W coincides as a $\mathbb{C}$ space with V, but is considered as an $\mathbb{H}$ space by defining kx to be Kx, as in (18).

In case IV_{-+} we can, as we now demonstrate, define a sesquilinear form $b\colon W \times W \to \mathbb{H}$ by

$$b(x, y) = x \cdot y + k^{-1}(Kx \cdot y). \tag{28}$$

In order to prove that b is $\mathbb{H}$ linear in its first variable it suffices to check the two properties

$$b(\mathrm{i}x, y) = \mathrm{i}b(x, y) \tag{29a}$$

$$b(kx, y) = kb(x, y). \tag{29b}$$

Property (29a) is immediate and (29b) is quickly checked:

$$\begin{aligned} b(kx, y) &= Kx \cdot y + k^{-1}K^2 x \cdot y \qquad \text{by (18) and (28)} \\ &= Kx \cdot y + k(x \cdot y) = kb(x, y). \end{aligned}$$

Now b enjoys the symmetry property

$$b(x, y)^{\theta} = b(y, x) \tag{30}$$

where θ denotes the opposite involutary automorphism of $\mathbb{H}$ defined by

$$q^{\theta} = k\bar{q}k^{-1}. \tag{31}$$

Upon noting that $\lambda^{\theta} = \lambda$ and $\lambda k = k\bar{\lambda}$, for $\lambda \in \mathbb{C} \subset \mathbb{H}$ and that $k^{\theta} = -k = k^{-1}$, we have

$$\begin{aligned} b(x, y)^{\theta} &= (x \cdot y)^{\theta} + (Kx \cdot y)^{\theta} k \\ &= x \cdot y + k(\overline{Kx \cdot y}) \\ &= x \cdot y - k(x \cdot Ky) \qquad \text{since } \tilde{K} = K^{-1} = -K \\ &= y \cdot x + k^{-1} Ky \cdot x = b(y, x). \end{aligned}$$

Consequently b is '$\mathbb{H}^{\theta}$ bilinear' in the sense that (in addition to being bi-additive) it satisfies

$$b(px, qy) = pb(x, y)q^{\theta} \qquad p, q \in \mathbb{H}. \tag{32}$$

Moreover the non-degeneracy of $x \cdot y$ implies that of $b(x, y)$.

It follows from (6), (9) and (19) that the $D(g)$, when viewed as the $\mathbb{H}$ linear operators $A(g)$ on W, lie inside the isometry group $\mathrm{O}(W) = \mathrm{O}(V) \cap \mathrm{GL}(W) = \mathrm{U}(V) \cap \mathrm{GL}(W)$ of the $\mathbb{H}$ space W (the latter being equipped with the non-degenerate $\mathbb{H}^{\theta}$ bilinear form b):

Case IV_{-+}. $D(g) = A(g)^{\mathbb{C}}$ where $A(g) \in \mathrm{O}(W) \cong \mathrm{O}(m; \mathbb{H})$. (33)

Here $\mathrm{O}(m; \mathbb{H})$ denotes the group of $m \times m$ quaternionic matrices which leave invariant the form

$$x^1(y^1)^{\theta} + x^2(y^2)^{\theta} + \ldots + x^m(y^m)^{\theta}$$

on $\mathbb{H}^m$—the group isomorphism $\mathrm{O}(W) \cong \mathrm{O}(m; \mathbb{H})$ coming about since, using (32), we have

$$b(x, y) = \sum_{a=1}^{m} x^a (y^a)^{\theta} \tag{34}$$

relative to any orthonormal basis $\{e_1, \ldots e_m\}$ for W: $b(e_a, e_b) = \delta_{ab}$. (Such bases exist: see, for example, Dieudonné (1971) and Porteous (1969).)

The case $\mathrm{IV}_{--}^{(r,s)}$ can be dealt with in a very similar fashion, upon defining

$$h(x, y) = (x, y) + k^{-1}(Kx, y). \tag{35}$$

We quickly check, using $K^{-1} = -K = K^{+}$, that h is a Hermitian form on W, in that (in addition to being bi-additive) it satisfies

$$\overline{h(x, y)} = h(y, x) \tag{36}$$

and

$$h(px, qy) = ph(x, y)\bar{q} \qquad p, q \in \mathbb{H}. \tag{37}$$

It follows that the $D(g)$, when viewed as the $\mathbb{H}$ linear operators $A(g)$ on W, lie inside the isometry group $\mathrm{Sp}(W) = \mathrm{Sp}(V) \cap \mathrm{GL}(W) = \mathrm{U}(V) \cap \mathrm{GL}(W)$ of the $\mathbb{H}$ space W (the latter being equipped with the non-degenerate Hermitian form h):

Case $IV_{--}^{(r,s)}$. $D(g) = A(g)^{\mathbb{C}}$ where $A(g) \in \mathrm{Sp}(W) \cong \mathrm{Sp}(p, q)$. (38)

Here, instead of (34) we have

$$h(x, y) = x^1\overline{y^1} + \ldots + x^p\overline{y^p} - x^{p+1}\overline{y^{p+1}} - \ldots - x^m\overline{y^m} \tag{39}$$

relative to an orthonormal basis $\{e_1, \ldots e_m\}$ for W of signature (p, q), $(p+q=m)$, and $\mathrm{Sp}(p, q)$ denotes the group of $m \times m$ quaternionic matrices which leave invariant the Hermitian form on $\mathbb{H}^m$ occurring on the RHS of (39). The signature (p, q) is of course an invariant of h; indeed, the basis $\{e_1, \ldots, e_m, Ke_1, \ldots, Ke_m\}$ for V will have, by (35), signature $(2p, 2q)$, whence $p = r/2$, $q = s/2$. Note, as previously announced, that r and s are necessarily even in the cases when $\varepsilon = \eta = -1$.

3. Summary

The foregoing natural classification of the irreducible complex representations of a group into various 'classical' types is summarised in table 1. In column 4 we give the

Table 1. Assignment of a complex n-dimensional irreducible group representation D to one† of ten classical types.

Case	Extra structure present	Subcases	Associated classical group	Restrictions on n and/or (r, s)
O	—	O	$\mathrm{GL}(n; \mathbb{C})$	—
I	Non-degenerate invariant bilinear form $x \cdot y = \eta y \cdot x$, $\eta = \pm 1$	$\mathrm{I}_\eta = \begin{cases} \mathrm{I}_+ \\ \mathrm{I}_- \end{cases}$	$\mathrm{O}(n; \mathbb{C})$ $\mathrm{Sp}(n; \mathbb{C})$	— $n = 2m$
II	Non-degenerate invariant Hermitian form (x, y) of signature (r, s)	$\mathrm{II}^{(r,s)}$	$\mathrm{U}(r, s)$	—
III	Structure map K $K^2 = \varepsilon I$, $\varepsilon = \pm 1$	$\mathrm{III}_\varepsilon = \begin{cases} \mathrm{III}_+ \\ \mathrm{III}_- \end{cases}$	$\mathrm{GL}(n; \mathbb{R})$ $\mathrm{GL}(m; \mathbb{H})$	— $n = 2m$
IV	All 3 structures $x \cdot y$, (x, y), K are present and are related by $(x, y) = \begin{cases} Kx \cdot y & \text{if } \varepsilon = \eta \\ iKx \cdot y & \text{if } \varepsilon = -\eta \end{cases}$	$\mathrm{IV}^{(r,s)}_{\varepsilon\eta} =$ $\mathrm{IV}^{(r,s)}_{++}$ IV_{+-} IV_{-+} $\mathrm{IV}^{(r,s)}_{--}$	$\mathrm{O}(r, s)$ $\mathrm{Sp}(n; \mathbb{R})$ $\mathrm{O}(m; \mathbb{H})$ $\mathrm{Sp}(p, q)$	— $n = 2m$, $r = s = m$ $n = 2m$, $r = s = m$ $n = 2m$, $r = 2p$, $s = 2q$

† The assignment will be unique if, by appropriate choice of the overall sign of (x, y), we arrange for the signature to satisfy $r \geqslant s$.

classical group, G say, which preserves the structures listed in column 2. Thus (at least if we take the carrier space V to be $\mathbb{C}^n$) in cases I and II the operators $D(g)$ form a subgroup of the stated classical group $\mathrm{G} \subset \mathrm{GL}(n; \mathbb{C})$, while in cases III and IV the operators $A(g)$ (upon $\mathbb{R}^n$ or upon $\mathbb{H}^m$), where $A(g)^{\mathbb{C}} = D(g)$, form a subgroup of the stated classical group $\mathrm{G} \subset \mathrm{GL}(n; \mathbb{R})$ or $\mathrm{G} \subset \mathrm{GL}(n/2; \mathbb{H})$.

Of course, groups of $m \times m$ quaternionic matrices can be thought of as groups of $2m \times 2m$ complex matrices, corresponding to the use of bases of the kind $\{e_1, \ldots, e_m, Ke_1, \ldots, Ke_m\}$ for the complex space $V = W^{\mathbb{C}}$, with $\{e_1, \ldots, e_m\}$ a basis for the quaternionic space W. When $GL\ (m; \mathbb{H})$ and $\mathrm{O}(m; \mathbb{H})$ are thought of in this way they are commonly denoted by $\mathrm{U}^*(2m)$ and $\mathrm{O}^*(2m)$ (see, for example, Helgason (1978) or Barut and Raczka (1977)).

Incidentally, as far as the classical orthogonal groups themselves are concerned, these are usually classified into two kinds according to whether n is odd or even, corresponding to their association with the complex simple Lie algebra B_k, where $2k+1=n$, or with D_k, where $2k=n$, respectively.

Bearing this notation in mind, the classical groups in table 1 are precisely those studied in the classical mathematical literature, and which can be found listed in table 11.53 of Porteous (1969), in chapter 3, § 7 of Barut and Raczka (1977), in chapter 10 of Helgason (1978), and in many other texts.

Altogether, discounting the signature aspect of the classification which enters into cases II, IV_{++} and IV_{--}, we obtain ten types of representation corresponding to the ten† classical groups listed in column 4. As discussed above, for the last three types listed the signature is wholly or partly restricted in value, as in column 5 of the table.

As a simple illustration of the above classification, consider the representation $D^{j,j'}$ of the group $\mathscr{G}=\mathrm{SL}\,(2;\mathbb{C})$. Bearing in mind that the basic representation $D^{1/2,0}$ is symplectic (since $\mathrm{SL}(2;\mathbb{C})=\mathrm{Sp}(2;\mathbb{C})$), that the complex conjugate of $D^{j,j'}$ is $D^{j',j}$, and that $D^{j,j'}$ is linearly equivalent to $D^{j',j}$ if and only if $j=j'$, the classification is as follows. If $j\neq j'$ then $D^{j,j'}$ is of type I_+ or I_- according to whether $2\,(j+j')$ is even or odd, while $D^{j,j}$ is of type IV_{++}, the signature being indefinite for $j\neq 0$.

Similarly, in the case of the group SU(2) the representation $D^{1/2}$ is both unitary and, since $\mathrm{SU}(2)\subset\mathrm{SL}(2;\mathbb{C})=\mathrm{Sp}(2;\mathbb{C})$, symplectic, and so is of type $\mathrm{IV}^{\mathrm{def}}_{--}$, where 'def' denotes that the signature is positive definite. Consequently the representation D^j of SU(2), being the $(2j)$th symmetrised tensorial power of $D^{1/2}$, is of type $\mathrm{IV}^{\mathrm{def}}_{++}$ or $\mathrm{IV}^{\mathrm{def}}_{--}$ according to whether $2j$ is even or odd.

Once one knows that a representation falls under case IV one should be immediately attuned to the possibility of making good use of any, or all, of the three structures $x\cdot y$, (x,y), K which are present. For example, when dealing with the $3j$-coefficients for the group SU(2) it helps to stress the bilinear form $x\cdot y$ rather than the Hermitian form (x,y) (see Shaw (1983), § 12.3).

4. The Frobenius–Schur types

In terms of the above classification these are seen to subdivide as follows:

$$\begin{aligned}
&(0)\quad \text{strictly complex:} && \mathrm{O}\cup\mathrm{I}\cup\mathrm{II}\\
&(+)\quad \text{real:} && \mathrm{III}_+\cup\mathrm{IV}_{++}\cup\mathrm{IV}_{+-}\\
&(-)\quad \text{quaternionic:} && \mathrm{III}_-\cup\mathrm{IV}_{-+}\cup\mathrm{IV}_{--}.
\end{aligned}\tag{40}$$

Let us specialise now to a representation D which is pseudo-unitary, i.e. falls under case II or IV. *Let us further suppose that the signature* (r,s) *is non-neutral, i.e.* $r\neq s$. This last supposition rules out cases IV_{+-} and IV_{-+}, leaving only the three possibilities

$$D\in\mathrm{II}^{(r,s)}\cup\mathrm{IV}^{(r,s)}_{++}\cup\mathrm{IV}^{(r,s)}_{--}\tag{41}$$

which are associated with the respective classical groups $\mathrm{U}(r,s)$, $\mathrm{O}(r,s)$, $\mathrm{Sp}(r/2,s/2)$. Upon comparing (40) and (41) we obtain the following result: *if D is any irreducible,*

† Porteous (1969, p 270) (see also pp 207 and 216) assigns code numbers 0, 1, 2, . . . , 9 to these ten classical groups; in the order in which they are listed in the table, their Porteous codes are 9, 7, 3, 8, 1, 5, 0, 2, 6, 4.

pseudo-unitary representation for which the signature is non-neutral, then

$$D \text{ is} \begin{cases} \text{strictly complex} \\ \text{real} \\ \text{quaternionic} \end{cases} \Leftrightarrow D \text{ is} \begin{cases} \text{inequivalent to } \hat{D} \\ \text{orthogonal} \\ \text{symplectic.} \end{cases} \tag{42}$$

In particular this result applies to an irreducible representation which is unitary in the strict sense (i.e. positive definite signature) and so holds whenever the group $\mathscr{G}$ is finite or compact. This particular case of (42) is of course well known (see, for example, Adams (1969), theorem 3.50).

In the case of a finite or compact group $\mathscr{G}$ there is (see, for example, Adams (1969), theorem 3.62) a well known classical criterion, involving averaging over the group, which decides the Frobenius-Schur type (ε), $\varepsilon = 0, +, -$, of an irreducible representation D, namely

$$Av_{g\in\mathscr{G}}\chi(g^2) = \varepsilon \tag{43}$$

where $\chi(g) = \operatorname{Tr} D(g)$. It may be of interest to give a proof of this classical result (43) in the present setting. To this end let us make use of the following form of the group orthogonality relations for D:

$$AvD(g)x \cdot D(g^{-1})y = n^{-1}y \cdot x \tag{44}$$

valid (see Shaw (1983), (13.1.41)) for any bilinear form $x \cdot y$. If D falls under case IV, then we can take $x \cdot y$ to be the previously considered invariant bilinear form, whence (44) leads to

$$AvD(g^2)x \cdot y = \eta n^{-1}x \cdot y$$

and hence

$$Av_{g\in\mathscr{G}}D(g^2) = \varepsilon n^{-1}I. \tag{45}$$

Here we have used the fact (see (41)) that $\eta = \varepsilon$ for a (strictly unitary) representation $D \in \mathrm{IV}$. Upon taking the trace, we obtain the desired group character result (43) for the cases $\varepsilon = +1$ or $\varepsilon = -1$. To obtain (43) in the case $\varepsilon = 0$ when $D \in \mathrm{II}$, simply use the group orthogonality relations for the inequivalent (see (42)) representations D and $\hat{D}$.

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